Day-2 lecture-2

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Lecture outline:

(1) Entropy — density of states — further holographic checks — central charge and Newton's constant.

(2) Rest of holographic RG.

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We have seen in the homework that the entropy of the $AdS_5$-schwarzchild geometry is given by $S = \frac{A}{4G_N} = \frac{\pi^2 r_+^3}{2G_N}$,

In 3-dimensional bulk the relevant metric is that of the BTZ black hole $ds^2 = \left(\frac{r^2}{\ell^2} - 8M\right)dt^2 + \frac{dr^2}{\frac{r^2}{\ell^2} - 8M} + r^2 d\phi^2$ and this gives entropy

$$S = \frac{A}{4G_N} = \frac{2\pi r_+}{4G_N} = \boxed{\frac{\ell \pi^2}{G_N \beta}}$$

Using $\beta = \frac{4\pi}{f'(r_+)} = \frac{4\pi \ell}{2r_+}$

$$r_+ = \frac{2\pi \ell}{\beta}$$

The 3d BTZ at inverse temperature $\beta$ describes a thermal state of a CFT on a circle. Thus the Euclidean spacetime of the CFT is on a torus.

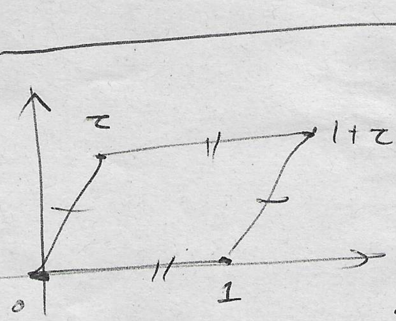
$$Z = \mathrm{Tr}\, e^{-\beta H} \underset{\beta \to \infty}{\sim} e^{-\beta E_0} = e^{\frac{\beta c}{12}}$$


$\leftarrow \tau = \frac{i\beta}{2\pi}$

A consequence of conformal symmetry in 2 dimensions is modular invariance

$$\tau \to \frac{a\tau + b}{c\tau + d}$$

$$\tau \to -\frac{1}{\tau}$$




$$\beta = \frac{2\pi\tau}{i} \leftarrow \tau = \frac{i\beta}{2\pi}$$

$$\beta_{new} = -\frac{2\pi}{i\tau} = -\frac{2\pi 2\pi}{i\, i\beta} = \frac{4\pi^2}{\beta}$$

So when $\beta \to \infty$ (low T), $\beta_{new} \to 0$ (high T)

$$\therefore \quad Z \overset{\beta \to 0}{\sim} e^{\frac{c\pi^2}{3\beta}}$$

$$S = \left(1 - \beta \partial_\beta\right) \log Z$$

$$= \frac{c\pi^2}{3\beta} + \cancel{\beta} \frac{c\pi^2}{3\cancel{\beta^2}\beta}$$

$$= \frac{2c\pi^2}{3\beta} = \frac{\ell \pi^2}{\beta G_N}$$

This tells us that $c = \dfrac{3\ell}{2G_N}$

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This ~~...~~ result is consistent with the conformal anomaly ~~...~~ calculations of the boundary CFT. ie; under $g_{(0)} \to e^{2\sigma} g_{(0)}$

$$\delta g_{(0)} = g_{(0)} 2 \delta\sigma$$

$$\delta W_{CFT} = \int d^2x \, A \, \delta\sigma \sim c$$

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Same result can be calculated from the bulk:

$$W_{CFT} \underset{AdS/CFT}{=} e^{-S_{\text{on-shell}}[g_{(0)}]} \qquad -\frac{1}{16\pi G_N} \int (R + 2\Lambda) + \cdots$$

$$\delta W_{CFT} \equiv e^{-S_{\text{onshell}}[g_{(0)} + \delta g_{(0)}]} - e^{-S_{\text{on-shell}}[g_{(0)}]}$$

$$\sim \frac{1}{G_N} \cdots$$

(3)

Some words on Cardy formula

$$r_+ = \ell\sqrt{8M} = \ell\sqrt{\frac{8EG_N}{\ell}}$$

S could also be expressed as

$$S = \frac{2\pi\ell}{4G_N}\sqrt{8M}$$

$$= \frac{\pi\ell}{G_N}\sqrt{2M}$$

$$= \frac{\cancel{\ell}\,\pi\,2c}{3\ell}\sqrt{2M}$$

$$= \frac{\pi\ell}{G_N}\sqrt{\frac{2EG_N}{\ell}}$$

$$= \frac{\pi\cancel{\ell}\,2c}{3\cancel{\ell}}\sqrt{\frac{\cancel{2}E}{\ell}\frac{3\cancel{\ell}}{\cancel{2}c}}$$

$$\boxed{S = 2\pi\sqrt{\frac{c}{3}E}}$$

$$G_N = \frac{3\ell}{2c}$$

$$E = \frac{\ell M}{G_N}$$

$$\frac{E}{\ell} = \frac{2Mc}{3}$$

$$M = \frac{3E}{2c}$$

$$= \frac{3E}{2\frac{3}{2}\frac{\ell}{G_N}}$$

$$= \frac{E}{\ell}G_N.$$

This is consistent with Cardy formula which estimate the degeneracy of high energy states:

start with low $\beta$ expansion:

$$\int \rho(E)\,dE\,e^{-\beta E} = e^{\frac{c\pi^2}{3\beta}}$$

$$\therefore \rho(E) = \int d\beta\, e^{\beta E}\, e^{\frac{c\pi^2}{3\beta}}$$

$$\simeq e^{2\pi\sqrt{\frac{cE}{3}}}$$

$$E = -\partial_\beta \log Z.$$

$$= \frac{M\ell}{G_N}.$$

$$\therefore M = \frac{E\,G_N}{\ell}$$

$$E = \frac{c\pi^2}{3\beta_s^2}$$

$$\beta_s = \sqrt{\frac{c}{3E}}\,\pi$$

$$\therefore \beta_s E + \frac{c\pi^2}{3\beta_s}$$

$$= \pi\sqrt{\frac{cE}{3}} + \frac{c\pi}{3}\sqrt{\cdots}$$

$$= 2\pi\sqrt{\frac{cE}{3}}$$

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(2) Holographic RG 1010.1264v2

$$Z = \int \partial \tilde{\phi} \; \Psi_{IR} \, \Psi_{UV}$$

where: $\Psi_{IR} = \Psi_{IR}(\tilde{\phi}) = \int \partial \phi \, e^{-S}$

$$\Psi_{UV} = \Psi_{UV}(\tilde{\phi}, \phi_\epsilon)$$

$$\frac{dZ}{dl} = 0$$

$$Z = \langle \Psi_{UV} | \, e^{-\int_{+\infty}^{\epsilon} \mathcal{H} dz} \, | \Psi_{IR} \rangle$$

Inserting complete set of states

$$= \int \partial \tilde{\phi} \, \langle \Psi_{UV} | \, e^{-\int_{l}^{\epsilon} \mathcal{H} dz} \, | \tilde{\phi} \rangle \langle \tilde{\phi} | \, e^{-\int_{\infty}^{l} \mathcal{H} dz} \, | \Psi_{IR} \rangle$$

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$$0 = \frac{dZ}{dl} = \int \partial \tilde{\phi} \left\{ \langle \Psi_{UV} | \left( \mathcal{H} e^{-\int_{l}^{\epsilon} \mathcal{H} dz} | \tilde{\phi} \rangle + e^{-\int_{l}^{\epsilon} \mathcal{H} dz} \partial_l | \tilde{\phi} \rangle \right) \right\}$$

$$\left\{ \left( \partial_l \langle \tilde{\phi} | + \langle \tilde{\phi} | - \mathcal{H} \, e^{-\int_{\infty}^{l} \mathcal{H} dz} \right) | \Psi_{IR} \rangle \right\}$$

$$\Rightarrow \quad \partial_l \Psi_{IR} = \mathcal{H} \Psi_{IR} \quad \text{and} \quad \boxed{\partial_l \Psi_{UV} = -\mathcal{H} \Psi_{UV}}$$

$$\mathcal{H} \Psi_{UV}^* + \partial_l \Psi_{UV}^* = 0$$

Let $\Psi_{uv} = e^{-F}$

$$\mathcal{H} = \frac{1}{2\ell}\left(\ell^d \tilde{\pi}^2 + \frac{1}{\ell^d} m^2 \tilde{\phi}^2\right)$$

$$\mathcal{H}\Psi_{uv} = \frac{1}{2\ell}\left(\ell^d\left(\frac{-\delta^2}{\delta\tilde{\phi}^2}\right) + \frac{1}{\ell^d} m^2 \tilde{\phi}^2\right) e^{-F}$$

$$\frac{\delta^2}{\delta\tilde{\phi}^2} e^{-F} = \frac{\delta}{\delta\tilde{\phi}}\left(\left(-\frac{\delta F}{\delta\tilde{\phi}}\right) e^{-F}\right)$$

$$= \left[-\frac{\delta^2 F}{\delta\tilde{\phi}^2} + \left(\frac{\delta F}{\delta\tilde{\phi}}\right)^2\right] e^{-F}$$

$$= \frac{1}{2\ell}\left[\ell^d \frac{\delta^2 F}{\delta\tilde{\phi}^2} - \ell^d\left(\frac{\delta F}{\delta\tilde{\phi}}\right)^2 + \frac{1}{\ell^d} m^2 \tilde{\phi}^2\right] \Psi_{uv}$$

$$\partial_\ell \Psi_{uv} = -\partial_\ell F \, \Psi_{uv}$$

$$\therefore \quad \partial_\ell \Psi_{uv} = -H \Psi_{uv}$$

$$\Rightarrow \partial_\ell F = \frac{1}{2\ell}\left[\ell^d \frac{\delta^2 F}{\delta \tilde{\phi}^2} - \ell^d \left(\frac{\delta F}{\delta \tilde{\phi}}\right)^2 + \frac{1}{\ell^d} m^2 \tilde{\phi}^2\right]$$

with

$$F = \frac{1}{2h(\ell)} \int d^d x \, \frac{1}{\ell^d} \left(\tilde{\phi} + g(\ell)\right)^2$$

R.H.S

$$= \frac{1}{2\ell}\Bigg[\ell^d \frac{1}{2h} \frac{1}{\ell^d} 2 \left(\cancel{\tilde{\phi} + g(\ell)}\right)$$

$$- \ell^d \frac{1}{h^2} \frac{1}{\ell^{2d}} \left(\tilde{\phi} + g\right)^2 + \frac{1}{\ell^d} m^2 \tilde{\phi}^2\Bigg]$$

$$= \frac{1}{2h\ell} - \frac{1}{2\ell \ell^d h^2} \left(\tilde{\phi} + g\right)^2 + \frac{m^2 \tilde{\phi}^2}{2\ell \ell^d}$$

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L.H.S

$$= \tilde{\phi}^2 \left( - \frac{\ell^{-d} h'}{2h^2} - \frac{d \ell^{-1-d}}{2h} \right)$$

$$+ \tilde{\phi} \left( - \frac{d \ell^{-1-d}}{h} g + \frac{\ell^{-d} g'}{h} - \frac{\ell^{-d} g h'}{h^2} \right) + \cdots$$

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<u>Comparing coefficients yield:</u>

(a) $\tilde{\phi}^2$ coeff:

$$\ell \partial_\ell h = 1 - dh - m^2 h^2 \quad — (A)$$

$\tilde{\phi}$ coeff using (A)

$$\ell g' = -m^2 g h \quad — (B)$$

fixed pt of (A)

$$h_* = \frac{1}{-2m^2}\left(d \pm \sqrt{d^2 + 2m^2}\right)$$

$$= -\frac{1}{m^2}\left(\frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2}\right)$$

$$= -\frac{1}{m^2}\left(\frac{d}{2} \pm \nu\right) = -\frac{\Delta_\pm}{m^2}.$$

plugging into (B):

$$\ell \partial_\ell g = + m^2 g \frac{\Delta_\pm}{m^2}$$

$$\therefore \ell \partial_\ell g = g \Delta_\pm$$

$$\therefore \log g \sim \Delta_\pm \log \ell$$

$$\therefore g \sim \ell^{\Delta_\pm}$$

$$\Rightarrow [O] \Rightarrow \Delta_\mp \checkmark.$$

And this

$$e^{-S} = \int \mathcal{D}\tilde{\phi}\; \Psi_{IR}\, \Psi_{UV}$$

$$= \int \mathcal{D}\tilde{\phi}\; \mathcal{D}M\; e^{-S_0[M] + \int \frac{d^dx}{\ell^d}\left(\tilde{\phi}\,\mathcal{O} - \frac{1}{2h}(\tilde{\phi}+g)^2\right)}$$

$$S = S_0[M] + \int \frac{d^dx}{\ell^d}\left(g\,\mathcal{O} - \frac{1}{2} h\, \mathcal{O}^2\right).$$

So $h$ is the double trace deformation and $g$ is the coupling of $\mathcal{O}$.

~~If we~~ Under these pert:

$$f_k^1 = \frac{f_0}{1+f_0 G_0}.$$

$$\Rightarrow \beta_f = -2\nu f + 2\nu b f^2$$

If we study (A) around $h = h_*$

$$h = h_* + \delta h.$$

$$\ell\,\partial_\ell(\delta h) = 1 - d\left(h_* + \delta h\right) - m^2\left(h_* + \delta h\right)^2$$

$$= \cancel{1} - d\left(\cancel{-\frac{1}{m^2}\left(\frac{d}{2} \pm \nu\right)} + \delta h\right) - m^2\left(\cancel{-\frac{1}{m^2}\left(\frac{d}{2} \pm \nu\right)} + \delta h\right)^2$$

$$= \cancel{1 - dh_* + m^2 h_*^2} - d\,\delta h - 2m^2 h_*\,\delta h - m^2 \delta h^2$$

$$= -d\,\delta h - 2m^2\,\delta h\left(-\frac{\Delta_\pm}{m^2}\right) - m^2\delta h^2$$

$$\Delta_\pm = \frac{d}{2} \pm \nu$$

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$$\ell \partial_\ell (\delta h) = -m^2 (\delta h)^2 + (\delta h)\left(2\Delta_\pm - d\right)$$

$$= -m^2 (\delta h)^2 \pm (\delta h)\, 2\nu .$$

$$\therefore \quad \beta_{\delta\lambda} = -\ell \partial_\ell (\delta h) = \mp 2\nu (\delta h) + m^2 (\delta h)^2 .$$

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Reproduced !

# Discussion on conformal anomaly using holography



We saw the example of AdS-schwarzschild in the homework note that as we get closer to bndy we get asymptotically AdS. And holography is supposed to work in asympt. AdS spacetimes.

We saw
$$Z[\phi_0]_{gravity} = \int_{\phi_0} \mathcal{D}\phi_{bulk} \exp\left(-S[\phi_{bulk}]_X\right)$$

and

$\phi_{bulk}$ is dual to $\mathcal{O}$ living on the CFT in $M = \partial X$. Bndy cond of $\phi_{bulk}$ is $\phi_0$

$$Z[\phi_0]_{CFT} = \left\langle \exp \int_M d^dx\, \mathcal{O}\phi_0 \right\rangle$$

When the gravity is weakly coupled then
$$Z[\phi_0]_{gravity} \sim \exp\left(-S[\phi_{cl}(\phi_0)]\right).$$

Another important field left out in last lecture is the bulk metric : $G_{\mu\nu}$. This is dual to the boundary stress tensor $T_{ij}$. The boundary data of $G_{\mu\nu}$ is <u>not</u> the bndy metric $(g_{(0)})_{ij}$ but only the conformal structure $[g_{(0)}]$. Clear from tutorials: ie; $g_{(0)} \sim \exp(2\sigma(x))\, g_{(0)}$. The conformal structure $[g_{(0)}]$ along with Einstein's equations specifies the metric $G_{\mu\nu}$. Thus naturally we get a CFT on $M = \partial X$.

However the quantum effective action $W_{CFT}[g_{(0)}]$ is not quantumly conformal symmetric.
ie; $W_{CFT}[g_{(0)}]$ is not invariant under $\delta g_{(0)} = 2\delta\sigma\, g_{(0)}$ but transforms as :
$$\delta W_{CFT}[g_{(0)}] = \int_M d^dx \sqrt{\det g_{(0)}}\, \mathcal{A}\, \delta\sigma$$
where $\mathcal{A}$ is the anomaly.

e.g in 2D CFT $\mathcal{A} = -\frac{c}{24\pi} R$

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In the CFT side the conformal anomaly $\mathcal{A}$ is related to $\langle T^{\mu}{}_{\mu}\rangle_{CFT}$ because:
$$Z = \int \mathcal{D}\phi\, e^{-S}$$
under $g_{(0)} \to (1 + 2\delta\sigma) g_{(0)}$ $\quad \delta g_{ij} = 2\delta\sigma\, g_{ij}$

$$\delta S = \int \frac{\delta S}{\delta g_{\mu\nu}} \delta g_{\mu\nu} = \int T^{ij} \delta g_{ij} = \int 2\delta\sigma\, T^{\mu}{}_{\mu}$$

$$\therefore \delta W_{CFT} \sim \int \delta\sigma \langle T^{\mu}{}_{\mu}\rangle_{CFT}.$$

In 2 dimensions:

Now when we use AdS/CFT to calculate $\delta W_{CFT}$ from the bulk, our starting point is $Z[\phi_0]_{grav} = \exp\left(-S_{\text{on-shell}}[\phi_{cl}[\phi_0]]\right)$

steps: 1 $G_{\mu\nu}\, dx^\mu dx^\nu = \frac{\ell^2}{4} \rho^{-2} d\rho\, d\rho + \rho^{-1} g_{ij}\, dx^i dx^j$

$g$ has limit $g_{(0)}$ as one goes to the boundary $\rightarrow \rho = 0$.

2 Solve Einstein's equations order by order in $\rho$

3 Find regulated action evaluated on the boundary:

$$W[g_{(0)}] = \frac{1}{16\pi G_N} \int d^dx \sqrt{|g_{(0)}|} \left( \epsilon^{-d/2} a_{(0)} + \epsilon^{-d/2+1} a_{(2)} + \dots + \epsilon^{-1} a_{(d-2)} - \log \epsilon\, a_{(d)} \right) + W_{finite}[g_{(0)}]$$

where the coefficients $a_{(\cdot)}$ are made out of $g_{(0)}$.

4 $\rightarrow$ We are after $\delta W_{finite}[g_{(0)}]$ under $\delta g_{(0)} = 2\delta\sigma\, g_{(0)}$.

All the contribution in the div. piece comes from the logarithmic term.

$$\delta W_{finite} = -\frac{1}{16\pi G_N} \int d^dx \sqrt{|g_{(0)}|}\; 2\, a_{(d)}\, \delta\sigma$$

$$\therefore\; A = -\frac{1}{16\pi G_N} 2\, a_{(d)}.$$

For 2 dim

$$a_{(d=2)} = \frac{\ell}{2} R(g_{(0)})$$

$$\therefore\; \boxed{A = -\frac{c}{24\pi} R}$$

$$\boxed{c = \frac{3\ell}{2G_N}}$$

$$-\frac{3\ell}{2G_N\, {}_{8}24\pi} R$$

$$-\frac{1}{8\pi G_N} \frac{\ell R}{2}$$

In particular calculating $SW_{CFT}$ from the bulk will always give term proportional to $G_N^{-1}$ While calculating the conformal anomaly from the CFT we find the anomaly to be related to central charge.

Thus we see that $G_N^{-1} \sim c \Rightarrow$ gravity is weakly coupled when the central charge is large (# d.o.f is large).

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For more discussion look at : https://arxiv.org/abs/hep-th/9806087